

Derivatives with respect to the order of the Legendre Polynomials

Bernard J. Laurenzi

Department of Chemistry, UAlbany, The State University of New York
1400 Washington Ave., Albany, N. Y. 12222

December 28, 2014

Abstract

Expressions for the derivatives of the Legendre polynomials of the first kind with respect to the order of these polynomials i.e. $P_n(z) = [\partial^n P_\nu(z)/\partial \nu^n]_{\nu=0}$ are given. An explicit form for the fourth derivative is presented.

1 Introduction

Recently, R. Szmytkowski [1] has obtained expressions for the first three derivatives of the Legendre functions of the first kind $P_\nu(z)$ with respect to order ν i.e. $P_n(z) = [\partial^n P_\nu(z)/\partial \nu^n]_{\nu=0}$ for $1 \leq \nu \leq 3$. That is to say,

$$\begin{aligned} P_0(z) &= 1, \\ P_1(z) &= \ln\left(\frac{1+z}{2}\right) = -Li_1\left(\frac{1-z}{2}\right), \\ P_2(z) &= -2Li_2\left(\frac{1-z}{2}\right), \\ P_3(z) &= 12Li_3\left(\frac{1+z}{2}\right) - 6\ln\left(\frac{1+z}{2}\right)Li_2\left(\frac{1+z}{2}\right) - \pi^2\ln\left(\frac{1+z}{2}\right) - 12\zeta(3), \end{aligned}$$

where $Li_\mu(z)$ is the polylogarithm function [2] of order μ and $\zeta(s)$ is the Riemann zeta function. These derivatives arise in studies of tidal hydrodynamics and are part of a recent and continuing interest in the variation of well known polynomials and other higher transcendental functions with respect to their orders [3],[4],[5],[6],[7].

In this note we will give a general expression for the derivatives $P_n(z)$ and discuss the expected increasing complexity of these expressions as n increases.

The Legendre functions of the first kind $P_\nu(z)$ satisfy the differential equation [8]

$$\left[\frac{d}{dz}(1-z^2)\frac{d}{dz} + \nu(\nu+1)\right]P_\nu(z) = 0, \quad -1 \leq z \leq 1.$$

Differentiating the latter expression with respect to ν and evaluating the result at $\nu = 0$ we get for $P_n(z)$ the relation

$$\frac{d}{dz} \left[(1 - z^2) \frac{d P_n(z)}{dz} \right] = -n P_{n-1}(z) - n(n-1) P_{n-2}(z).$$

Expressions for the desired derivatives $P_n(z)$ can then be reduced to quadratures by

$$P_n(z) = -n \int \frac{dz}{1 - z^2} \int^z [P_{n-1}(z') + (n-1) P_{n-2}(z')] dz' + C \ln\left(\frac{1+z}{1-z}\right), \quad (1)$$

where C is a constant of integration. The inner or *first* integrals in (1) will be discussed further below. With the use of the identity

$$P_\nu(1) = 1,$$

from which it follows that for $n \geq 1$

$$P_n(1) = 0,$$

the constants of integration which arise in (1) can be evaluated.

2 The Expression for $P_4(z)$

In this case we have to evaluate the expression

$$P_4(z) = C \ln\left(\frac{1+z}{1-z}\right) - 4 \int \frac{dz}{1 - z^2} \int^z [P_3(z') + 3 P_2(z')] dz',$$

or more explicitly

$$P_4(z) = C \ln\left(\frac{1+z}{1-z}\right) - 4 \int \frac{I(z)}{1 - z^2} dz,$$

where

$$I(z) = \int^z [12 Li_3\left(\frac{1+z'}{2}\right) - 6 \ln\left(\frac{1+z'}{2}\right) Li_2\left(\frac{1+z'}{2}\right) - \pi^2 \ln\left(\frac{1+z'}{2}\right) - 12 \zeta(3) - 6 Li_2\left(\frac{1-z'}{2}\right)] dz'.$$

The integrals in $I(z)$ are well known and we find the somewhat remarkable result

$$I(z) = (z+1)[12 Li_3\left(\frac{1+z}{2}\right) - 6 \ln\left(\frac{1+z}{2}\right) Li_2\left(\frac{1+z}{2}\right) - \pi^2 \ln\left(\frac{1+z}{2}\right) - 12 \zeta(3)].$$

In the outer i.e. *second* integration we have for $P_4(z)$

$$P_4(z) = C \ln\left(\frac{1+z}{1-z}\right) - 4 \int \frac{dz}{1 - z^2} [12 Li_3\left(\frac{1+z}{2}\right) - 6 \ln\left(\frac{1+z}{2}\right) Li_2\left(\frac{1+z}{2}\right) - \pi^2 \ln\left(\frac{1+z}{2}\right) - 12 \zeta(3)],$$

where C is a constant of integration. All but one of the integrals are elementary. With a change of variable

$$t = \frac{1+z}{2}, \quad 0 \leq t \leq 1$$

we get for $P_4(z)$

$$P_4(z) = C \ln\left(\frac{t}{1-t}\right) + C' + 24 \left[Li_2(t)^2 + 2 \ln(1-t) [Li_3(t) - \zeta(3)] + \frac{\pi^2}{6} Li_2(1-t) + \mathfrak{I}(t) \right], \quad (2)$$

where

$$\mathfrak{I}(t) = \int \frac{\ln(t) Li_2(t)}{1-t} dt,$$

and C' is the second constant of integration.

The integral $\mathfrak{I}(t)$ has been obtained within *Mathematica* [9]. After some simplification we have

$$\begin{aligned} \mathfrak{I}(t) = & [\ln^2(t) - \ln(t) \ln(1-t)] Li_2(t) - \ln^2(1-t) Li_2(1-t) + \ln^2\left(\frac{t}{1-t}\right) Li_2\left(\frac{t}{t-1}\right) - \frac{1}{2} Li_2(t)^2 \\ & + 2 \ln(t) Li_3(t) + 2 \ln(1-t) Li_3(1-t) - 2 \ln\left(\frac{t}{1-t}\right) Li_3\left(\frac{t}{t-1}\right) \\ & + 2 \left[Li_4(t) - Li_4(1-t) + Li_4\left(\frac{t}{t-1}\right) \right] + \ln^2(1-t) \left[\frac{1}{2} \ln^2(t) - \ln(t) \ln(1-t) + \frac{1}{4} \ln^2(1-t) \right]. \end{aligned} \quad (3)$$

Using the following identities for the di and trilogarithm functions

$$Li_2(1-z) = -Li_2(z) + \pi^2/6 - \ln(z) \ln(1-z),$$

$$Li_2\left(\frac{z}{z-1}\right) = -Li_2(z) - \frac{1}{2} \ln^2(1-z),$$

$$Li_3\left(\frac{z}{z-1}\right) + Li_3(1-z) + Li_3(z) - \zeta(3) = \frac{\pi^2}{6} \ln(1-z) - \frac{1}{2} \ln(z) \ln^2(1-z) + \frac{1}{6} \ln^3(1-z),$$

and gathering terms in equations 2 and 3 we get

$$P_4(z) = \pi^4/15 + 24 \left[\begin{aligned} & \frac{1}{2} Li_2(t)^2 - \frac{\pi^2}{6} Li_2(t) + 2 Li_4(t) - 2 Li_4(1-t) + 2 \ln(t) \{ Li_3(1-t) - \zeta(3) \} \\ & + \frac{1}{12} \ln^4(1-t) + \frac{\pi^2}{6} \ln^2(1-t) + 2 Li_4\left(\frac{t}{t-1}\right) \\ & + \ln(t) \ln(1-t) \{ Li_2(t) - \frac{\pi^2}{2} - \frac{1}{3} \ln^2(1-t) + \ln(t) \ln(1-t) \} \end{aligned} \right].$$

The constants C and C' in equation 4 having been evaluated by setting $P_4(1) = 0$ with the result that $C = 0$ and $C' = \pi^4/15$. We note that the expression in equation 4 contains the complicated term $Li_4\left(\frac{t}{t-1}\right)$. In contrast to the corresponding expressions for the polylogarithms Li_2 and Li_3 with the same argument, this term does not appear to be able to be rewritten in terms of the polylogarithm Li_4 with simpler arguments as noted by Lewin [10]. As a consequence it does not bode well for any likelihood of obtaining explicit analytic expressions for $P_n(z)$ for $n \geq 5$.

Appendix A

Each of the *first* integrals i.e.

$$\mathcal{I}_\eta(z) = \int^z \mathcal{P}_\eta(z') dz',$$

appearing in equation 1 occur twice i.e. in successive calculations of the quantities $\mathcal{P}_n(z)$ and $\mathcal{P}_{n+1}(z)$. The first few of these quantities are given below. It should be noted however, that these expressions are more complicated than the combinations

$$\int^z [\mathcal{P}_{n-1}(z') + (n-1) \mathcal{P}_{n-2}(z')] dz'.$$

This is due to an internal cancellation of terms within the combinations. The latter circumstance may indicate a deeper issue involving the $\mathcal{I}_\eta(z)$ integrals. We have

$$\begin{aligned} \mathcal{I}_1(z) &= (1+z) \left[\ln\left(\frac{1+z}{2}\right) - 1 \right], \\ \mathcal{I}_2(z) &= -2(1+z) \left[\ln\left(\frac{1+z}{2}\right) - 1 \right] + 2(1-z) \operatorname{Li}_2\left(\frac{1-z}{2}\right), \end{aligned}$$

$$\begin{aligned} \mathcal{I}_3(z) &= 6(1+z) \left[2 \operatorname{Li}_3\left(\frac{1+z}{2}\right) + \frac{\pi^2}{6} - 1 + 2\zeta(3) - \left\{ \operatorname{Li}_2\left(\frac{1+z}{2}\right) + \frac{\pi^2}{6} - 1 \right\} \ln\left(\frac{1+z}{2}\right) \right] \\ &\quad + 6(1-z) \left[\operatorname{Li}\left(\frac{1+z}{2}\right) + \ln\left(\frac{1-z}{2}\right) \ln\left(\frac{1+z}{2}\right) \right], \end{aligned}$$

The expression for $\mathcal{I}_4(z)$ has been computed using *Mathematica* and contains in a condensed form seventy-four terms including a fifth order polylogarithm function. That quantity will not be displayed here for the sake of brevity.

Below we include integrals which occur in the calculation of $\mathcal{I}_4(z)$ but are not directly available within *Mathematica* i.e.

$$\begin{aligned} \int \operatorname{Li}_4\left(\frac{t}{t-1}\right) dt &= -\frac{1}{2} \operatorname{Li}_2\left(\frac{t}{t-1}\right)^2 + t \operatorname{Li}_4\left(\frac{t}{t-1}\right) + \ln(1-t) \operatorname{Li}_3\left(\frac{t}{t-1}\right), \\ \int \operatorname{Li}_2(t)^2 dt &= -2 + 6t + 6\left[1-t - \frac{\pi^2}{9}\right] \ln(1-t) - 2[1-t - \ln(t)] \ln^2(1-t) \\ &\quad - 2[t - (1+t) \ln(1-t)] \operatorname{Li}_2(t) + t \operatorname{Li}_2(t)^2 + 4 \operatorname{Li}_3(1-t), \end{aligned}$$

$$\begin{aligned} \int \ln^2(t) \ln^2(1-t) dt &= \\ &-4 + 24x + 12[1-x] \ln(1-x) - 2[1-x] \ln(1-x)^2 - \frac{1}{2} \ln(1-x)^4 \\ &- 12x \ln(x) - 4[1-2x] \ln(1-x) \ln(x) - 2x \ln(1-x)^2 \ln(x) + 2 \ln(1-x)^3 \ln(x) \\ &+ [2 - \ln(1-x)^2] \ln(x)^2 - (1-x)[2 - 2 \ln(1-x) + \ln(1-x)^2] \ln(x)^2 \\ &+ [4 - 4 \ln(1-x) + 2 \ln(1-x)^2] \operatorname{Li}_2(1-x) - [4 - 4 \ln(x) + 2 \ln(x)^2] \operatorname{Li}_2(x) \\ &- [2 \ln(1-x)^2 - 4 \ln(1-x) \ln(x) + 2 \ln(x)^2] \operatorname{Li}_2\left(\frac{x}{x-1}\right) \\ &- 4[1 - \ln(x)] \operatorname{Li}_3(x) + 4[1 - \ln(1-x)] \operatorname{Li}_3(1-x) + 4[\ln(x) - \ln(1-x)] \operatorname{Li}_3\left(\frac{x}{x-1}\right) \\ &+ 4 \operatorname{Li}_4(1-x) - 4 \operatorname{Li}_4(x) - 4 \operatorname{Li}_4\left(\frac{x}{x-1}\right). \end{aligned}$$

Appendix B

The integral $\mathfrak{I}(z)$ is interesting in that it provides a way to obtain a closed form expression for the slowly converging infinite sum $\sum_{k=1}^{\infty} \frac{\Psi'(k)}{k^2}$ where $\Psi'(k)$ is the trigamma function [11]. This sum does not appear to have been previously reported in the literature and is included here. Using the limiting values for $\mathfrak{I}(z)$ i.e.

$$\begin{aligned}\mathfrak{I}(1) &= -\frac{11}{360}\pi^4, \\ \mathfrak{I}(0) &= -\frac{1}{45}\pi^4,\end{aligned}$$

and the infinite series representation for the dilogarithm function which occurs in $\mathfrak{I}(z)$ we get

$$\begin{aligned}\mathfrak{I}(1) - \mathfrak{I}(0) &= \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^1 \frac{z^k \ln(z)}{1-z} dz, \\ -\frac{\pi^4}{120} &= -\sum_{k=1}^{\infty} \frac{\Psi'(k+1)}{k^2}.\end{aligned}$$

Expanding the trigamma function with the use of the recurrence relation for Ψ' i.e.

$$\Psi'(k+1) = \Psi'(k) - 1/k^2,$$

together with the well known sum $\sum_{k=1}^{\infty} 1/k^4 = \pi^4/90$ we get the value of the desired summation i.e.

$$\sum_{k=1}^{\infty} \frac{\Psi'(k)}{k^2} = \frac{7}{360}\pi^4.$$

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